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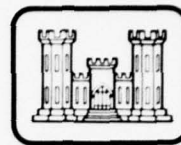UNCLASSIFIED   CRREL-79-27   NL
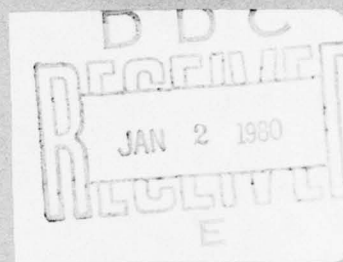
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$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_0\left(\beta e^{\frac{\pi i}{2}}\sqrt{x^2+y^2}\right) e^{i\xi x + i\eta y}\, dx\, dy = \frac{-4i}{\xi^2 + \eta^2 + \beta^2}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\xi^2 + \eta^2 + \beta^2} e^{-ix\xi - iy\eta}\, d\xi\, d\eta = \pi^2 i H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{x^2+y^2}\right)$$

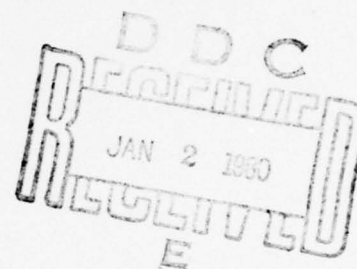79 12 31 034

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Some Bessel function identities arising in ice mechanics problems

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Shunsuke Takagi

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November 1979

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Prepared for
DIRECTORATE OF MILITARY PROGRAMS
OFFICE, CHIEF OF ENGINEERS
By
UNITED STATES ARMY
CORPS OF ENGINEERS
COLD REGIONS RESEARCH AND ENGINEERING LABORATORY
HANOVER, NEW HAMPSHIRE, U.S.A.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER CRREL Report 79-27	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SOME BESSEL FUNCTION IDENTITIES ARISING IN ICE MECHANICS PROBLEMS		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Shunsuke Takagi		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS U.S. Army Cold Regions Research and Engineering Laboratory Hanover, New Hampshire 03755		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS DA Project 4A161102AT24 Scientific Area A, Work Unit 005
11. CONTROLLING OFFICE NAME AND ADDRESS Directorate of Military Programs Office, Chief of Engineers Washington, D.C. 20314		12. REPORT DATE November 1979
		13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

DDC
RECEIVED
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RECEIVED
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18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Bessel functions
Fourier integrals
Ice mechanics
Winkler foundations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
Some Bessel function identities found by solving problems of the deflection of a floating ice plate by two different methods are rigorously proved. The master formulas from which all the identities are derived are in a Fourier reciprocal relationship, connecting a Hankel function to an exponential function. Many new formulas can be derived from the master formulas. The analytical method presented here now opens the way to study a hitherto impossible type of problem—the deflection of floating elastic plates of various shapes and boundary conditions.

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PREFACE

SOME BESSEL FUNCTION IDENTITIES ARISING IN ICE MECHANICS PROBLEMS

Shunsuke Takagi

INTRODUCTION

By solving several problems of the deflection of a floating ice plate (i.e. the deflection of a plate on a continuous elastic foundation as formulated by Winkler 1867) by two different methods, Kerr (1977a, 1978) presented a number of equality relationships among Bessel functions. In this report, the analytical derivation of his formulas is presented. His formulas (six in all) reduce to the following *two master formulas* expressed in the Fourier reciprocal relationship:

$$\int_{-\infty}^{\infty} e^{ix\xi} \, H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{a^2+\xi^2}\right)d\xi = \frac{2}{i\sqrt{x^2+\beta^2}} e^{-a\sqrt{x^2+\beta^2}}, \tag{1}$$

$$\int_{-\infty}^{\infty} e^{-a\sqrt{x^2+\beta^2}\,-\,i\xi x} \, \frac{dx}{\sqrt{x^2+\beta^2}} = \pi i \, H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{a^2+\xi^2}\right), \tag{2}$$

where $H_0^{(1)}(\ )$ is the Hankel function of zeroth order; x and ξ are real and a nonnegative such that $a^2+\xi^2 \neq 0$; and β is complex such that $\beta \neq 0$, $|\arg\beta| < \pi/2$, $x^2+\beta^2 \neq 0$. By $\sqrt{z}$ we mean such a branch as $\mathrm{Re}\sqrt{z} \geqq 0$. The master formulas may be transformed to double integrals that are symmetric with regard to arguments:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_0\left(\beta e^{\frac{\pi i}{2}}\sqrt{x^2+y^2}\right)e^{i\xi x + i\eta y} \, dxdy = \frac{-4i}{\xi^2+\eta^2+\beta^2}, \tag{3}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\xi^2+\eta^2+\beta^2} e^{-ix\xi - iy\eta} \, d\xi d\eta = \pi^2 i \, H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{x^2+y^2}\right). \tag{4}$$

In the following, we prove the above formulas, transform the two master formulas to derive all the formulas introduced by Kerr (1977a, 1978) as well as some new ones, and finally show that the analytical method presented here lays the foundation for building mathematical machinery for solving problems for various shapes of floating ice plates under various boundary conditions.

PROOF OF FORMULA (1)

Use of Barnes' integral representation of $H_0^{(1)}(z)$,

$$\pi i \, H_0^{(1)}(z) = \frac{1}{2\pi i}\int_{-c-\infty i}^{-c+\infty i} \Gamma^2(-s)\left(-\frac{iz}{2}\right)^{2s} ds, \tag{5}$$

where c is any positive number and $|\arg(-iz)| < \pi/2$ (Watson 1962, p. 192), transforms the single integral on the left-hand side of eq 1,

$$I_1 = \pi i \int_{-\infty}^{\infty} e^{ix\xi}\, H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{a^2+\xi^2}\right) d\xi, \tag{6}$$

to the repeated integrals

$$I_1 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{ix\xi}\, d\xi \int_{-c-\infty i}^{-c+\infty i} \Gamma^2(-s)\left(\frac{\beta}{2}\sqrt{a^2+\xi^2}\right)^{2s} ds. \tag{7}$$

The absolute convergence of the integral (5) carries over to (7), because the condition $|\arg(-iz)| < \pi/2$ in (5) transforms to $|\arg\beta| < \pi/2$ in (7), which is one of the prerequisites in the master formulas. Therefore, the order of integration in (7) may be exchanged. Moreover, restricting the original range of a, which is $a \geq 0$, to $a > 0$, we let $\xi = a\eta$ in (7) so that it becomes

$$I_1 = \frac{a}{2\pi i}\int_{-c-\infty i}^{-c+\infty i} \Gamma^2(-s)\left(\frac{\beta a}{2}\right)^{2s} ds \int_{-\infty}^{\infty} e^{iax\eta}(1+\eta^2)^s\, d\eta. \tag{8}$$

To evaluate the internal single integral in (8)

$$M_1 = \int_{-\infty}^{\infty} e^{iaxz}(1+z^2)^s\, dz$$

by the contour integral method, we first note that the original range of x (i.e. $-\infty < x < \infty$) may be restricted to $0 < x$ (where the case $x = 0$ is excluded), because I_1 is an even function of x, as the right-hand side of (6) shows. Consider the contour in Figure 1 that starts at origin 0, goes along the positive real axis to A (i.e. $z = \infty$), takes a 90° turn along the infinitely large circle to reach B (i.e. $z = i\infty$), comes down along the imaginary axis to C (i.e. $z = i$), makes a 360° turn along an infinitely small circle clockwise around C, goes upward along the imaginary axis to reach D (i.e. $z = i\infty$), takes a 90° turn along the infinitely large circle to reach E (i.e. $z = -\infty$), and finally reaches origin 0, thus completing a circuit. No singularity of the integrand $\exp(iaxz)(1+z^2)^s$ exists inside this closed contour. Among the integrals along the paths mentioned above, the integrals along AB and DE are equal to zero, provided that $a > 0$. The integral around C is also equal to zero. Therefore, on the condition that $a > 0$ and $x > 0$, we have

$$M_1 = -\left\{\int_{B}^{C_1} + \int_{C_2}^{D}\right\} e^{iaxz}(1+z^2)^s\, dz,$$

where C_1 and C_2 are the initial and terminal points of the infinitely small circle around C. Letting

Figure 1. Transformation of integral M_1 on the z-plane.

$z = it$, where t is real, M_1 reduces to

$$M_1 = i(1 - e^{-2\pi is}) \int_1^\infty e^{-axt} (1 - t^2)^s \, dt.$$

We now let s be

$$s = -\tfrac{1}{2} + ip;$$

i.e. let c in (5) be $\tfrac{1}{2}$, where p is a real number, in order to integrate M_1 by use of the formula in Watson (1962, p. 172),

$$K_\nu(z) = \frac{\Gamma(\tfrac{1}{2}) \, (\tfrac{1}{2}z)^\nu}{\Gamma(\nu + \tfrac{1}{2})} \int_1^\infty e^{-zt}(t^2 - 1)^{\nu - \tfrac{1}{2}} \, dt, \tag{9}$$

which is valid when $R_e(\nu + \tfrac{1}{2}) > 0$ and $|\arg z| < \pi/2$. These two conditions are satisfied when we let $\nu - \tfrac{1}{2} = s$ and $z = ax$ to integrate M_1. Thus, letting

$$(1 - t^2)^s = e^{\pi is} (t^2 - 1)^s,$$

M_1 integrates to

$$M_1 = -2 \sin \pi s \, \frac{\Gamma(s+1)}{\sqrt{\pi} \left(\frac{ax}{2}\right)^{s + \tfrac{1}{2}}} K_{s + \tfrac{1}{2}} (ax).$$

In this way, (7) transforms to a single integral:

$$I_1 = -\frac{a}{\pi i} \int_{-\tfrac{1}{2} - \infty i}^{-\tfrac{1}{2} + \infty i} \Gamma^2(-s) \left(\frac{\beta a}{2}\right)^{2s} \sin \pi s \, \frac{\Gamma(s+1)}{\sqrt{\pi} \left(\frac{ax}{2}\right)^{s + \tfrac{1}{2}}} K_{s + \tfrac{1}{2}}(ax) \, ds.$$

Changing the Gamma function of the negative argument to the positive argument by the reflection formula,

$$\Gamma(-s) = \frac{-\pi}{\Gamma(1 + s) \sin \pi s},$$

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I_1 becomes

$$I_1 = -\frac{1}{\pi i}\sqrt{\frac{2\pi a}{x}} \int_{-\frac{1}{2}-\infty i}^{-\frac{1}{2}+\infty i} \frac{\pi}{\sin \pi s}\frac{K_{s+\frac{1}{2}}(ax)}{\Gamma(s+1)}\left(\frac{\beta^2 a}{2x}\right)^s ds.$$

Taking the residues, I_1 integrates to

$$I_1 = \frac{8\pi a}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{\beta^2 a}{2x}\right)^n K_{n+\frac{1}{2}}(ax).$$

Replacing $K_{n+\frac{1}{2}}(ax)$ with

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \exp\left(-t - \frac{z^2}{4t}\right)t^{-\nu-1}\, dt,$$

i.e., a formula found in [8, p. 183], I_1 becomes

$$I_1 = q\sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\infty \left(\frac{a^2\beta^2}{4t}\right)^n \exp\left(-t - \frac{a^2 x^2}{4t}\right)\frac{dt}{\sqrt{t^3}}.$$

The order of the summation and integration may be exchanged, and we get

$$I_1 = q\sqrt{\pi} \int_0^\infty \exp\left(-t - \frac{a^2(x^2+\beta^2)}{4t}\right)\frac{dt}{\sqrt{t^3}}.$$

Letting $t = \xi^{-2}$, this becomes

$$I_1 = 2a\sqrt{\pi} \int_0^\infty \exp\left(-\xi^{-2} - \frac{a^2(x^2+\beta^2)}{4}\xi^2\right) d\xi. \tag{10}$$

To integrate (10), we introduce a lemma:

$$\int_0^\infty e^{-\mu^2\xi^2 - \xi^{-2}}\, d\xi = e^{-2\mu}\frac{\sqrt{\pi}}{2\mu},$$

provided that

$$n\pi - \frac{\pi}{4} < \arg\mu < n\pi + \frac{\pi}{4}$$

where n is an integer.

When μ is in the above range, the above integral is convergent. To prove the lemma, we first note that the integral

$$L_1 = \int_0^\infty e^{-\mu^2\xi^2 - \xi^{-2}}\, d\xi$$

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transforms to $L_1 = e^{-2\mu} N_1,$ $\qquad\qquad$ (11)

where $N_1 = \int_0^\infty e^{-(\mu\xi - \xi^{-1})^2} d\xi.$ $\qquad\qquad$ (12)

Letting $\xi = 1/(\mu\eta)$

and changing the resulting contour $0 \sim \mu^{-1}\infty$ to $0 \sim \infty$, we get

$$N_1 = \int_0^\infty e^{-(\mu\eta - \eta^{-1})^2} \frac{d\eta}{\mu\eta^2}.$$ $\qquad\qquad$ (13)

Addition of (12) and (13) yields

$$2N_1 = \frac{1}{\mu} \int_0^\infty e^{-(\mu\xi - \xi^{-1})^2} \left(\mu + \frac{1}{\xi^2}\right) d\xi.$$

Letting

$$\mu\xi - \xi^{-1} = t,$$

we get

$$2N_1 = \frac{1}{\mu} \int_{-\infty}^{\mu\infty} e^{-t^2} dt.$$

Changing the range of integration to the one from $-\infty$ to $+\infty$, N_1 integrates to $N_1 = \pi/(2\mu)$. Substituting this value into (11), the lemma is proved.

Letting μ be

$$\mu = \frac{a}{2}\sqrt{x^2 + \beta^2}$$

in the lemma, (10) is integrated, because μ above is obviously in the range prescribed before. Thus, under the conditions $x \neq 0$ and $a \neq 0$, Formula (1) is proved. Applying the analytical continuation, the condition $a \neq 0$ is extended to the condition $a^2 + \beta^2 \neq 0$. Because the integral is convergent at $x = 0$, the condition $x \neq 0$ may be removed. The proof is thus completed.

PROOF OF FORMULA (2)

Although Formula (2) is the Fourier inverse of Formula (1), we show an independent proof in view of the importance of the formula.

On the assumption that $a^2 + \xi^2 \neq 0$ and $\beta \neq 0$, letting

$$\begin{aligned} a &= r \cos\alpha \\ \xi &= r \sin\alpha \\ x &= \beta \operatorname{Sinh} z \\ -\frac{\pi}{2} &< \alpha < \frac{\pi}{2}, \end{aligned}$$ $\qquad\qquad$ (14)

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the integral

$$I_2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2 + \beta^2}} e^{-a\sqrt{x^2 + \beta^2} - i\xi x} \, dx \tag{15}$$

transforms to

$$I_2 = \int_L \exp\left(-r\beta \, \text{Cosh} \, (z + i\alpha)\right) dz, \tag{16}$$

where the contour L is a curve on the complex $z = u + i\nu$ plane (Fig. 2) defined by

$$z = \text{Arcsinh} \, (x/\beta)$$

$$= \log\left(\frac{x}{\beta} + \sqrt{\frac{x^2}{\beta^2} + 1}\right) \tag{17}$$

with parameter x in the range of $-\infty < x < \infty$.

Letting $x = 0$, we have $z = 0$. Therefore, contour L passes through the origin. When $x \rightarrow +\infty$ or $-\infty$, z asymptotically approaches $\log(2x) - \log \beta$, or $-\log(-2x) + \log \beta$, respectively; in other words, the imaginary part ν of complex variable z satisfies the conditions

$$\lim_{x \rightarrow +\infty} \nu + \arg \beta = 0$$

$$\lim_{x \rightarrow -\infty} \nu - \arg \beta = 0 \tag{18}$$

The curve L_+ defined for the case $0 < \arg \beta < \pi/2$ is shown in Figure 2. The curve L_- defined for the case $-\pi/2 < \arg \beta < 0$ is symmetrical with L_+ with regard to the real axis.

Letting u be the real part of z, we find that, as $x \rightarrow \infty$ or $-\infty$, the real part of $-r\beta\text{Cosh}\,(z + i\alpha)$ approaches asymptotically either

$$-\frac{1}{2} r|\beta| e^u \cos(\arg \beta + \nu + \alpha)$$

or

$$-\frac{1}{2} r|\beta| e^{-u} \cos(\arg \beta - \nu - \alpha),$$

respectively. Because the power of the exponent in (16) must remain negative as $|x| \rightarrow \infty$, the conditions

$$-\frac{\pi}{2} < \arg \beta + \lim_{x \rightarrow +\infty} \nu + \alpha < \frac{\pi}{2}$$

$$-\frac{\pi}{2} < \arg \beta - \lim_{x \rightarrow -\infty} \nu - \alpha < \frac{\pi}{2} \tag{19}$$

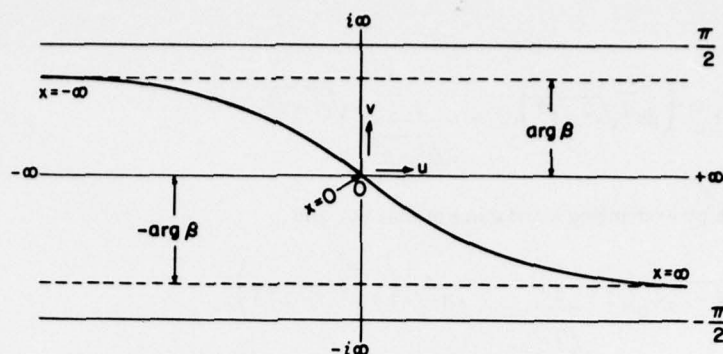must be satisfied. The conditions are satisfied by $(14)_4$ and (18).

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Figure 2. Curve L_+ defined for the case $0 < arg\beta < \pi/2$ in the complex $z = u + iv$ plane is shown for parameter x ranging over $-\infty < x < \infty$.

Introducing t defined by

$$t = z + i\alpha \tag{20}$$

I_2 becomes

$$I_2 = \int_{L + i\alpha} \exp(-r\beta \cosh t) \, dt.$$

The contour $L + i\alpha$ may be moved to the real axis, because doing this is tantamount to letting $\lim_{x \to \pm\infty} v + \alpha = 0$, which is evidently permissible in the prescribed range of β shown below (2). Thus we have

$$I_2 = \int_{-\infty}^{\infty} \exp(-r\beta \cosh t) \, dt. \tag{21}$$

To integrate I_2 in (21), we introduce the following formula in (Lebedev 1965). Provided $I_m (z) > 0$,

$$H_0^{(1)} (z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \exp(iz \cosh t) \, dt. \tag{22}$$

Comparing z in (22) with $-r\beta$ in (21) we have $I_m(z) = R_e (\beta r)$; therefore, the condition $I_m (z) > 0$ is satisfied in the prescribed range of β. Thus I_2 integrates to

$$I_2 = \pi i H_0^{(1)} \left(\beta e^{\frac{\pi i}{2}} r \right),$$

completing the proof of Formula 2.

TRANSFORMATIONS OF FORMULAS (1) AND (2)

We list the formulas found in Erdelyi et al. (1954) and Kerr (1978, 1977b) that can be derived from Formulas (1) and (2). The formulas we use in the following transformations are

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$$\int_{-\infty}^{\infty} e^{ia\xi} H_0^{(1)} \left(\beta e^{\frac{\pi i}{2}} \sqrt{x^2 + \xi^2} \right) d\xi = \frac{2}{i\sqrt{a^2 + \beta^2}} e^{-\sqrt{a^2 + \beta^2} x} \tag{23}$$

which is found by exchanging x and a in Formula (1), and

$$\int_{-\infty}^{\infty} e^{-a\sqrt{x^2 + \beta^2} - ibx} \frac{dx}{\sqrt{x^2 + \beta^2}} = \pi i H_0^{(1)} \left(\beta e^{\frac{\pi i}{2}} \sqrt{a^2 + b^2} \right), \tag{24}$$

which is found by changing ξ to b in Formula (2).

Assuming β to be a positive number, (23) transforms to a real integral:

$$\int_0^{\infty} K_0 \left(\beta \sqrt{x^2 + \xi^2} \right) \cos (a\xi) \, d\xi = \frac{\pi}{2\sqrt{a^2 + \beta^2}} e^{-\sqrt{a^2 + \beta^2} x}. \tag{25}$$

Letting $a = 0$, (25) may become

$$\int_{-\infty}^{\infty} K_0 \left(\beta \sqrt{x^2 + \xi^2} \right) d\xi = \frac{\pi}{\beta} e^{-\beta x}. \tag{26}$$

Letting $\xi = \eta - y$, and expressly specifying that a may be positive or negative, (23) may be re-written as

$$\int_{-\infty}^{\infty} e^{\pm ia\eta} H_0^{(1)} \left(\beta e^{\frac{\pi i}{2}} \sqrt{x^2 + (y - \eta)^2} \right) d\eta = \frac{2}{i\sqrt{a^2 + \beta^2}} e^{\pm ia\eta - \sqrt{a^2 + \beta^2} x} \tag{27}$$

We transform this to several forms. When β is real, (27) becomes

$$\int_{-\infty}^{\infty} e^{\pm ia\eta} K_0 \left(\beta \sqrt{x^2 + (y - \eta)^2} \right) d\eta = \frac{\pi}{\sqrt{a^2 + \beta^2}} e^{\pm iay - \sqrt{a^2 + \beta^2} x}. \tag{28}$$

Letting $\beta = b \exp (\pi i/4)$ with the restriction $b > 0$, (27) becomes

$$\int_{-\infty}^{\infty} e^{\pm ia\eta} \left\{ \ker \left(b\sqrt{x^2 + (y - \eta)^2} \right) + i \kei \left(b\sqrt{x^2 + (y - \eta)^2} \right) \right\} d\eta = \frac{\pi}{\sqrt{a^2 + ib^2}} e^{-\sqrt{a^2 + ib^2} x \pm iay}. \tag{29}$$

Letting

$$\sqrt{a^2 + ib^2} = p + iq \tag{30}$$

and adding and subtracting the plus expression and the minus expression in (29), we find two integrals:

$$\int_{-\infty}^{\infty} \cos a\eta \left\{ \ker \left(b\sqrt{x^2+(y-\eta)^2}\right) + i\ker \left(b\sqrt{x^2+(y-\eta)^2}\right) \right\} d\eta = \frac{\pi(p-iq)}{\sqrt{a^4+b^4}} e^{-(p+iq)x} \cos ay$$

$$\int_{-\infty}^{\infty} \sin a\eta \left\{ \ker \left(b\sqrt{x^2+(y-\eta)^2}\right) + i\ker \left(b\sqrt{x^2+(y-\eta)^2}\right) \right\} d\eta = \frac{\pi(p-iq)}{\sqrt{a^4+b^4}} e^{-(p+iq)x} \sin ay,$$

$$(31)$$

where

$$\left. \begin{array}{c} p \\ q \end{array} \right| = \frac{1}{\sqrt{2}} \sqrt{\sqrt{a^4+b^4} \pm a^2} \tag{32}$$

Dividing the above two formulas into real and imaginary parts, four real integrals may be found. Letting $\beta = 1$, (24) becomes

$$\int_{-\infty}^{\infty} e^{-a\sqrt{x^2+1} - ibx} \frac{dx}{\sqrt{x^2+1}} = 2 K_0(\sqrt{a^2+b^2}). \tag{33}$$

Differentiating (33) with regard to a, we find

$$\int_{-\infty}^{\infty} e^{-a\sqrt{x^2+1} - ibx} dx = -\frac{2a}{\sqrt{a^2+b^2}} K_0'(\sqrt{a^2+b^2}). \tag{34}$$

Formula (25) is listed by Erdély (1954) without proof. Formulas (25), (28) and (31)$_2$ were derived by Kerr (1978), and Formula (34) by Kerr (1977a). Because the differentiations and integrations with regard to parameters inside the integral, as well as the assignment of arbitrary values to the parameters, are permissible insomuch as the absolute convergence is preserved, many additional formulas may be derived from (23) and (24).

PROOF OF FORMULAS (3) AND (4)

Integrating once more the rewritten Formula (1)

$$\int_{-\infty}^{\infty} e^{i\xi x} H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}} \sqrt{x^2+y^2} \right) dx = \frac{2}{i\sqrt{\xi^2+\beta^2}} e^{-|y|\sqrt{\xi^2+\beta^2}},$$

where $|y|$ takes the place of the original positive number a, we transform the left-hand side of Formula (3),

$$I_3 = \int_{-\infty}^{\infty} e^{i\eta y} dy \int_{-\infty}^{\infty} e^{i\xi x} H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}} \sqrt{x^2+y^2} \right) dx,$$

to

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$$I_3 = \frac{2}{\sqrt{\xi^2+\beta^2}} \int_{-\infty}^{\infty} e^{i\eta y - |y|\sqrt{\xi^2+\beta^2}} \, dy.$$

Use of the integral

$$\int_{-\infty}^{\infty} e^{i\eta y - |y|\sqrt{\xi^2+\beta^2}} \, dy = \frac{2\sqrt{\xi^2+\beta^2}}{\xi^2+\eta^2+\beta^2} \tag{35}$$

reduces I_3 to the right-hand side of Formula (3). To prove (35), note that

$$\int_{-\infty}^{\infty} e^{i\eta y - |y|\sqrt{\xi^2+\beta^2}} \, dy = 2 \int_0^{\infty} e^{-y\sqrt{\xi^2+\beta^2}} \cos \eta y \, dy.$$

Although Formula (4) is the Fourier inverse of Formula (3), we show an independent proof in view of the importance of the formula. The rewritten Formula (2),

$$\int_{-\infty}^{\infty} e^{-|y|\sqrt{\xi^2+\beta^2} - ix\xi} \, \frac{d\xi}{\sqrt{\xi^2+\beta^2}} = \pi i \, H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}} \sqrt{x^2+y^2}\right),$$

reduces to Formula (4) by the substitution of the Fourier inverse of (35):

$$e^{-|y|\sqrt{\xi^2+\beta^2}} = \frac{\sqrt{\xi^2+\beta^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi^2+\eta^2+\beta^2} e^{-iy\eta} \, d\eta. \tag{36}$$

Formula (36) may otherwise be proved by showing that

$$\int_{-\infty}^{\infty} \frac{e^{-iyz}}{z^2+a^2} \, dz = \frac{\pi}{a} e^{-a|y|} \tag{37}$$

by use of the contour integral method, where $R_e \, a \neq 0$.

ADDITIONAL DERIVATION FROM THE MASTER FORMULAS

We prove the formula

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2+\xi^2}} H_{0,1}^{(1)}\left(\beta e^{\frac{\pi i}{2}} \sqrt{a^2+\xi^2}\right) H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}} \sqrt{x^2+(y-\xi)^2}\right) d\xi = \frac{2}{\beta a} H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}} \sqrt{(a+x)^2+y^2}\right),$$

$$\tag{38}$$

where

$$H_{0,1}^{(1)}(\lambda) = \left[\frac{d}{dz} H_0^{(1)}(z) \right]_{z=\lambda}. \tag{39}$$

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Differentiating (24) with regard to a, we find

$$\int_{-\infty}^{\infty} e^{-a\sqrt{x^2+\beta^2}-ibx} \, dx = \frac{\pi\beta a}{\sqrt{a^2+b^2}} H_{0,1}^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{a^2+b^2}\right). \tag{40}$$

Substitution of $H_{0,1}^{(1)}\left(\beta e^{\pi i/2}\sqrt{a^2+\xi^2}\right)$ from (40) transforms the single integral,

$$I_4 = \int_{-\infty}^{\infty} \frac{\pi\beta a}{\sqrt{a^2+\xi^2}} H_{0,1}^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{a^2+\xi^2}\right) H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{x^2+(y-\xi)^2}\right) d\xi, \tag{41}$$

to the repeated integrals,

$$I_4 = \int_{-\infty}^{\infty} H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{x^2+(y-\xi)^2}\right) d\xi \int_{-\infty}^{\infty} e^{-a\sqrt{t^2+\beta^2}-i\xi t} \, dt,$$

which, on changing the order of the integration, becomes

$$I_4 = \int_{-\infty}^{\infty} e^{-a\sqrt{t^2+\beta^2}} \, dt \int_{-\infty}^{\infty} e^{i\xi t} H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{x^2+(y-\xi^2)}\right) d\xi.$$

Letting $\xi = \eta - y$, and changing a to $-t$, (23) becomes

$$\int_{-\infty}^{\infty} e^{-it\eta} H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{x^2+(y-\eta)^2}\right) d\eta = \frac{2}{\sqrt{t^2+\beta^2}} e^{-ity-\sqrt{t^2+\beta^2}\,x}.$$

Using the last integral to carry out the internal integration, I_4 becomes

$$I_4 = \frac{2}{i}\int_{-\infty}^{\infty} \frac{1}{\sqrt{t^2+\beta^2}} e^{-(a+x)\sqrt{t^2+\beta^2}-ity} \, dt,$$

which integrates to

$$I_4 = 2\pi H_0^{(1)}\left(\beta e^{\frac{\pi i}{2}}\sqrt{(a+x)^2+y^2}\right) \tag{42}$$

by use of Formula (2). Combining (41) and (42), (38) is proved.
Letting $\beta = 1$ in (38) we find

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2+\xi^2}} K_0'\left(\sqrt{a^2+\xi^2}\right) K_0\left(\sqrt{x^2+(y-\xi)^2}\right) = -\frac{\pi}{a} K_0\left(\sqrt{(a+x)^2+y^2}\right). \tag{43}$$

This formula was derived by Kerr (1977a) with his indirect method.

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DEFLECTION OF THE FLOATING ICE PLATE

Expressed in nondimensional form, the differential equation governing the deflection w of a floating ice plate sustaining a concentrated load P at the origin ($x = 0, y = 0$) is

$$\nabla^4 w + w = P\delta(x)\,\delta(y), \tag{44}$$

where ∇^2 is the Laplacian operator and $\delta(\ )$ the delta function. The solution of (44) for an infinite plate is

$$w(x, y) = -\frac{P}{2\pi}\,\mathrm{kei}\sqrt{x^2 + y^2}, \tag{45}$$

as was shown by Wymann (1950) by examining the nature of the solution of the homogeneous form of (44). His proof, however, does not exactly show that (45) is the solution of the inhomogeneous equation (44). We now can prove this by direct substitution of (45) into (44).

Use of Formula (4) enables us to derive

$$\mathrm{kei}\sqrt{x^2 + y^2} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(\xi^2 + \eta^2)^2 + 1}e^{+ix\xi + iy\eta}\,d\xi d\eta. \tag{46}$$

With the use of (46), substitution of (45) reduces the left-hand side of (44) to

$$\frac{P}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{+ix\xi + iy\eta}\,d\xi d\eta,$$

which by the property of the delta function (Gel'fand and Shilov 1964) is the right-hand side of (44).

Use of the reciprocal Fourier relationships developed in this report opens the possibility of building analytical machinery for solving the generic equation

$$\nabla^4 w + w = P\,\delta(x - x_0)\,\delta(y - y_0) \tag{47}$$

for the deflection of floating plates of various shapes and of various boundary conditions, which has hitherto been impossible. (Currently only the image method is used; see Kerr (1963 or 1977b). It is especially encouraging to note that the solution $w(x, y)$ of (47) is a generalized function (Gel'fand and Shilov 1964).

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